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ELASTIC WAVES IN A MATERIAL WITH CHEMOMECHANICAL REACTIONS*

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A theoretical analysis is given of mechanical wave processes in muscle tissue over a broad frequency range. As in /1/, the elastic waves are studied using a continual chemomechanical model /2-5/ extended to the case of an arbitrary discrete and continuous relaxation time spectrum /6/. Analytic expressions containing elastic and viscous parameters, as well as parameters corresponding to the muscle anisotropy and activity, are obtained for the elastic wave velocity and damping in thin muscle tissue specimens. The muscle specimen stability conditions are found. A comparison is made with known experimental results and it is shown that the model constructed describes the elastic-wave characteristics satisfactorily in a muscle in different states.

Investigation of elastic-waves in a medium is an important (often unique) method of determining its structure and rheological and functional properties. This especially concerns media of a biological nature, particularly muscle and internal organ tissues. As a rule, biological media are anisotropic and heterogeneous, where the muscle tissue still manifest active properties, and develops a stress as a result of chemical reactions. During miscle contraction (single, say) the elastic-wave velocity and damping depend on the muscle stress and degree of contraction. Depending on the wavelength, the excitation method, and the propagation

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direction, mechanical waves of different types are possible in which the structural and rheological properties of the medium appear differently.

If the wavelength is small compared with the characteristic linear dimensions of the tissue specimen, longitudinal waves are possible that are due to the compressibility and propagate at different angles to the anisotropy axes, as are also transverse (shear) waves. As they apply to muscle tissue, these kinds of waves were examined in /1, 2, 7/. The longitudinal wave damping, unlike their velocity, depends strongly on the state of the muscle and the propgation direction (along or across the fibres) /1, 8/. Longitudinal /6/ and flexural displacement waves, as well as torsional waves are possible in specimens that are thin compared with the wavelength. Longitudinal and flexural waves are also possible in specimens in the form of thin plates whose thickness is small compared with the wavelength while the dimensions in the two other directions are large. Surface (Rayleigh) waves and some others are also of interest.

1. Muscle tissue is regarded as a biphasal and multicomponent continuous medium, where the active phase 1 is visoelastic and chemical reactions occur there. The passive phase 2 is elastic and sources exist there for substances needed for muscle activiation and contraction. Exchange of the substance can occur between the phases.

The coordinate system can be curvilinear. The stress tensor components σ^{ij} of the medium consist of stress tensor components $\sigma_1{}^{ij}$ of the active phase and $\sigma_2{}^{ij}$ of the passive phase taking their volume contents φ_1 and φ_2 into account:

$$\sigma^{ij} = \varphi_1\sigma_1{}^{ij} + \varphi_2\sigma_2{}^{ij} \quad (\varphi_1 + \varphi_2 = 1) \tag{1.1}$$

Let the deformations be small and the medium incompressible. In this case the rheological equations describing the muscle tissue have the form

$$\sigma_1{}^{ij} = p_1 g^{ij} + \sigma_{1\Sigma}^{ij}, \quad \sigma_{1\Sigma}^{ij} = \sum_{\alpha=1}^{\infty} \sigma_{1\alpha}^{ij} \tag{1.2}$$

$$\sigma_{1\alpha}^{ij} = p_\alpha' g^{ij} + L_\alpha^{ijkl}\frac{d\Delta_{kl}^\alpha}{dt} + N_\alpha^{ij} = A_\alpha^{ijkl}\eta_{kl}^\alpha$$

$$\sigma_2{}^{ij} = p_2 g^{ij} + B^{ijkl}\varepsilon_{kl}, \quad \varepsilon_{kl} = \eta_{kl}^\alpha + \Delta_{kl}^\alpha \tag{1.3}$$

The quantities ε_{kl} are strain tensor components of the medium as a whole, $\eta_{kl}{}^\alpha$ is the reversible part of the strain tensor component, and $\Delta_{kl}{}^\alpha$ is the irreversible part, $N_\alpha{}^{ij}$ are the active stress tensor components governed by the chemical reactions, and p_α', $p_{1,2}$ are Lagrange multipliers. It is assumed that $\Delta_{ii}{}^\alpha = 0$ and $\eta_{ii}{}^\alpha = 0$, and therefore, $\varepsilon_{ii} = 0$. The components A_α^{ijmn} and B^{ijkl} are elasticity coefficients while L_α^{ijkl} is the viscosity and g^{ij} are metric tensor components.

These equations are obtained by using the methods of the mechanics of a continuous medium and non-equilibrium thermodynamics as in /1-5/. The balance relations for the phase concentrations of the phase substances and the total energy of each phase, the theorem of vital phase forces, the heat influx equations of the phases, and Gibbs' identity for each phase are used here (where the internal energy of the active phase 1 depends on $\eta_{mn}^\alpha, \alpha = 1, 2, \ldots$). On the basis of these, an entropy balance equation is obtained for the medium, from which relations between the thermodynamic fluxes and forces result, in particular, the equation in (1.2) connecting $\sigma_{1\alpha}{}^{ij}$ with $d\Delta_{kl}{}^\alpha/dt$ and with $N_\alpha{}^{ij}$. Unlike /3-5/, where only slow stress and strain change processes were taken into account, the complete relaxation time spectrum is contained in the model (1.1)-(1.3), as is manifiest as the freqeuncy of action on the medium increases. In principle, by assuming the relaxation time spectrum to be continuous (see below), an integral can be written in place of the sum in (1.2).

It is assumed in the model (1.1)-(1.3) that a small material volume of the active phase 1 contains a discrete set (generally infinite) of subelements connected in parallel, with numbers α ($\alpha = 1, 2, \ldots$), each of which is characterized by the coefficients of viscosity L_α^{ijkl}, the coefficients of elasticity A_α^{ijmn} as well as the active stress $N_\alpha{}^{ij}$. The one-dimensional analogue of the model is shown in Fig.1. Phase 2 in it consists of the elastic element B, and phase 1 from subelements with elasticities $A_1, A_2, \ldots, A_\nu, \ldots$, viscosities $L_1, L_2, \ldots, L_\nu, \ldots$, and activities $N_1, N_2, \ldots, N_\nu, \ldots$ The parameters of the element B and the subelements with $A_\alpha, L_\alpha, N_\alpha$ ($\alpha = 1, 2, \ldots, \nu, \ldots$) change as a result of chemical reactions (during muscle activation, say). The simplest model of such a change in the presence of just three elements B, A_1, L_1 is proposed in /9/. A scheme analogous to Fig.1 was proposed in /10/ for instance for a passive model ($N_\alpha = 0$, $\alpha = 1, 2, \ldots, \nu, \ldots$).

The quantities $L_\alpha{}^{ijkl}$, $A_\alpha{}^{ijmn}$, B^{ijkl} are tensor coefficients determined by the metric tensor.

g^{ij} and the tensor b^{ij} characterizing the anisotropy. The hypothesis of transversal isotropy holds for a muscle, where the extracted direction agrees with the muscle fibre direction. In the case /11/

$$B^{ijkl} = \mu(g^{ik}g^{jl} + g^{il}g^{jk}) + \nu_1(g^{ik}b^{jl} + g^{jk}b^{il} + g^{il}b^{jk} + g^{jl}b^{ik}) + \nu_2 g^{ij}b^{kl} + \nu_3 b^{ij}b^{kl} \tag{1.4}$$

Here μ is the usual Lamé isotropic coefficient and $\nu_{1,2,3}$ define the transversal isotropy. Analogous expressions hold for A_α^{ijkl} ($\mu_A{}^\alpha$ $\nu^\alpha_{A1,2,3}$ will replace μ and $\nu_{1,2,3}$) and for L_α^{ijkl} ($\mu_L{}^\alpha$, $\nu^\alpha_{L1,2,3}$ replace μ, $\nu_{1,2,3}$). The coefficients B^{ijkl}, A_α^{ijkl}, L_α^{ijkl} depend, as does N_α^{ij}, on the state of the muscle tissue which can change because of the chemical processes.

In general, it can be required that the incompressibility condition is satisfied for each of the phases. In this case additional components and coefficients appear in the expressions for A_α^{ijkl} and L_α^{ijkl}. If the medium is compressible, then (1.1)-(1.3) hold for its description except that

$$\sigma_1^{ij} = \sum_{\alpha=1}^{\infty} \sigma_{1\alpha}^{ij}, \quad \sigma_{1\alpha}^{ij} = L_\alpha^{ijkl}\frac{d\Delta^\alpha_{kl}}{dt} + N_\alpha{}^{ij} = A_\alpha^{ijkl}\eta^\alpha_{kl}$$
$$\sigma_2^{ij} = B^{ijkl}\varepsilon_{kl}$$

will replace the first relations of (1.2) and (1.3), where the expressions for B^{ijkl}, $A_\alpha{}^{ijkl}$, $L_\alpha{}^{ijkl}$ will contain additional components as compared with (1.4) /1/, and $\varepsilon_{ii} \neq 0$, $\eta_{ii}{}^\alpha \neq 0$, $\Delta_{ii}{}^\alpha \neq 0$.

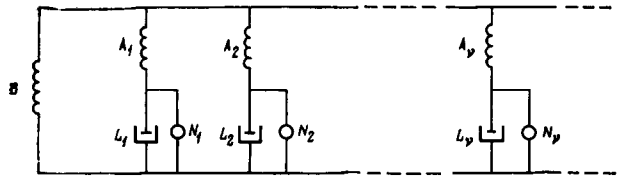

Fig.1

For simplicity, we can set $N_\alpha{}^{ij} = 0$ for $\alpha \geqslant 2$, $N_1{}^{ij} \neq 0$ and also $L_1{}^{ijkl} = 0$. We will take the active stress tensor components $N_1{}^{ij}$ in the form

$$N_1{}^{ij} = n_1{}^{ij} + m_1{}^{ij} \tag{1.5}$$

where $n_1{}^{ij}$ corresponds to the presence of the active stresses developed because of the biochemical reactions, and $m_1{}^{ij}$ reflects the contribution to the active stresses as a result of elastic deformation of the microrelations formed during the biochemical reactions (see Sect.5).

An expression for $n_1{}^{ij}$ is obtained from the condition that the number of muscle fibres in a specimen section is constant during deformation (freezing of the anisotropy) /2, 3/.

For finite deformations we have

$$n_1{}^{ij} = \gamma_1\sqrt{\frac{g_{33}}{G_{33}}}\,b^{ij}, \quad b^{ij} = \frac{1}{g_{33}}\frac{\partial x^i}{\partial \xi^3}\frac{\partial x^j}{\partial \xi^3} \tag{1.6}$$

$$dx^i = \left(\delta_k{}^i + \frac{\partial w^i}{\partial x^k}\right)d\xi^k, \quad G_{33} = 2\varepsilon_{33} + g_{33} \tag{1.7}$$

where x^i are Eulerian and ξ^i are Lagrangian coordinates, and the axis ξ^3 coincides with the direction of the muscle fibre, w^k are the displacement vector components of the medium, $\delta_k{}^i$ is the Kronecker delta, $\gamma_1 = \gamma_1(l_\Delta)$ is the activity parameter dependent on the degree of overlap of the active centres of the centres of the structural elements of the muscle tissue fibre l_Δ on which the following microrelations are formed:

$$l_\Delta = (1 + 2\varepsilon_{33}/g_{33})^{1/2} - (1 + 2\eta_{33}{}^1/g_{33})^{1/2} \tag{1.8}$$

The components g_{33}, G_{33}, ε_{33}, $\eta_{33}{}^1$ in (1.6)-(1.8) are taken in the ξ^i coordinate system and $n_1{}^{ij}$, b^{ij}, w^i in the x^i system.

If the deformations are small and the x^3 axis coincides with the fibre direction, then

$$n_1{}^{ij} = \frac{1}{g_{33}}\left(\gamma_1\delta_3{}^i\delta_3{}^j + \gamma_1\delta_3{}^i\frac{\partial w^j}{\partial x^3} + \gamma_1\delta_3{}^j\frac{\partial w^i}{\partial x^3} + \gamma_1'\delta_3{}^i\delta_3{}^j\Delta_{33}{}^1 - \gamma_1\frac{\varepsilon_{33}}{g_{33}}\delta_3{}^i\delta_3{}^j\right), \quad \gamma_1' = \frac{\partial\gamma_1}{\partial l_\Delta}, \quad l_\Delta = \frac{\Delta_{33}{}^1}{g_{33}} \tag{1.9}$$

$$m_1{}^{ij} = \Gamma_1^{ijkl}\Delta^1_{kl} \tag{1.10}$$

where Γ_1^{ijkl} defines the elastic properties of the microrelations. A formula of the type (1.4) holds for Γ_1^{ijkl} where $\Gamma_\mu{}^1$ and $\Gamma_{\nu_{1,2,3}}{}^1$ replace μ and $\nu_{1,2,3}$.

We rewrite (1.2) eliminating $\Delta^\alpha{}_{kl}$

$$L_\alpha^{abkl} A^{\circ\alpha}_{ijkl}\dot{\sigma}_{1\alpha}^{ij} + \sigma_{1\alpha}^{ab} = L_\alpha^{abkl}\varepsilon_{kl} + N_{\alpha p}^{ab} \qquad (1.11)$$
$$A^{\circ\alpha}_{ijkl} A_\alpha^{ijmn} = \delta_k{}^m\delta_l{}^n, \quad N_{\alpha p}^{ab} = N_\alpha^{ab} + p_\alpha' g^{ab}$$

Since $N_\alpha{}^{ab}$ depends on $\Delta_{kl}{}^\alpha$ and ε_{kl}, it is still necessary to append the relationship

$$\Delta^\alpha_{mn} = \varepsilon_{mn} - A^{\circ\alpha}_{ijmn}\sigma_{1\alpha}^{ij} \qquad (1.12)$$

For mechanical actions proceeding sufficiently rapidly the viscous properties of the muscle tissue do not succeed in appearing completely. For instance, let the displacement in the tissue occur at a frequency f. The characteristic times of component variation $\sigma_{1\alpha}{}^{ij}$ and ε_{kl} equal, respectively, the quantities $\tau_\alpha{}^\sigma(i, j)$ and $\tau_\alpha{}^\varepsilon(k, l)$ that are combinations of $L_\alpha{}^{ijkl}$, $A_\alpha{}^{ijkl}$, $N_\alpha{}^{ij}$ (see (1.5), (1.9)-(1.12)). If $f \gg 1/\tau_\alpha{}^\sigma(i, j)$, then (1.11) goes over into

$$L_\alpha^{abkl} A^{\circ\alpha}_{ijkl}\dot{\sigma}_{1\alpha}^{ij} = L_\alpha^{abkl}\dot{\varepsilon}_{kl} + N_{\alpha p}^{ab}$$

If $f \gg 1/\tau^\sigma{}_\alpha(i, j)$, $1/\tau_\alpha{}^\varepsilon(k, l)$, then (1.11) is simplified still more

$$\sigma_{1\alpha}^{ij} = A_\alpha^{ijkl}\varepsilon_{kl} \qquad (1.13)$$

Therefore, for a sufficiently high action frequency f the contribution of the active stresses $N_\alpha{}^{ij}$ can be neglected, and the distinction between the active and passive muscles will be contained in just the coefficients $A_\alpha{}^{ijkl}$, B^{ijkl}, $L_\alpha{}^{ijkl}$.

In the case when $f \ll 1/\tau_\alpha{}^\sigma(i, j)$ we have

$$\sigma_{1\alpha}^{ab} = L_\alpha^{abkl}\varepsilon_{kl} + N_{\alpha p}^{ab}$$

For $f \ll 1/\tau_\alpha{}^\sigma(i, j)$, $1/\tau_\alpha{}^\varepsilon(k, l)$ we obtain

$$\sigma_{1\alpha}^{ab} = N_{\alpha p}^{ab} \qquad (1.14)$$

Let the following conditions hold:

$$\ldots, \frac{1}{\tau^\sigma_{\nu-1}(i,j)}, \frac{1}{\tau^\varepsilon_{\nu-1}(k,l)} \ll f \sim \frac{1}{\tau_\nu{}^\sigma(i,j)} \sim \frac{1}{\tau_\nu{}^\varepsilon(k,l)} \ll \qquad (1.15)$$
$$\frac{1}{\tau^\sigma{}_1(i,j)}, \frac{1}{\tau^\varepsilon_{\nu+1}(k,l)}, \ldots$$

Then the subelements of the active phase 1 with the numbers $1, 2, 3, \ldots, \nu - 1$ can be described by (1.13), i.e., these subelements become purely elastic in practice. The subelements with numbers $\nu + 1, \nu + 2, \ldots$ can be described by (1.14), which reduce to trivial values when there are no active stresses. Therefore, under conditions (1.15) the equation connecting $\sigma_{1\Sigma}^{ij} = \sum_{\alpha=1}^{\infty}\sigma_{1\alpha}^{ij}$ with ε_{ij} has the form

$$L_\nu^{abkl} A^{\circ\nu}_{ijkl}\dot{\sigma}_{1\Sigma}^{ij} + \sigma_{1\Sigma}^{ab} = A_\Sigma^{abkl}\varepsilon_{kl} + L_\nu^{abkl}(A^{\circ\nu}_{ijkl} A_\Sigma^{ijmn} + \qquad (1.16)$$
$$\delta_k{}^m\delta_l{}^n\dot{\varepsilon}_{mn} + N_{\Sigma p}^{ab} + N_{\nu p}^{ab} + L_\nu^{abkl} A^{\circ\nu}_{ijkl} N_{\Sigma p}^{ij}$$
$$A_\Sigma^{abkl} = \sum_{\alpha=1}^{\nu-1} A_\alpha^{abkl}, \quad N_{\Sigma p}^{ab} = \sum_{\alpha=\nu+1}^{\infty} N_{\alpha p}^{ab}$$

If the above-mentioned conditions hold, namely $N_\alpha{}^{ij} = 0$ for $\alpha \geqslant 2$, $L_1{}^{ijkl} = 0$ and

$$A_{\Sigma 2}^{abkl} = \sum_{\alpha=2}^{\nu-1} A_\alpha^{abkl}, \quad p_\Sigma = \sum_{\alpha=\nu+1}^{\infty} p_\alpha'$$

then we obtain a simpler equation instead of (1.16)

$$L_\nu^{abkl} A^{\circ\nu}_{ijkl}\dot{\sigma}_{1\Sigma}^{ij} + \sigma_{1\Sigma}^{ab} = A_{\Sigma 2}^{abkl}\varepsilon_{kl} + L_\nu^{abkl}(A^{\circ\nu}_{ijkl} A_{\Sigma 2}^{ijmn} + \delta_k{}^m\delta_l{}^n)\dot{\varepsilon}_{mn} + \qquad (1.17)$$
$$N_{1p}^{ab} + p_\Sigma g^{ab} + L_\nu^{abkl} A^{\circ\nu}_{ijkl}(N_{1p}^{ij} + \dot{p}_\Sigma g^{ij})$$

Thus, at sufficiently high frequencies, the elastic properties of phase 1 are determined by the total elasticity coefficients of the subelements $1, 2, \ldots, \nu - 1$, and the viscous properties by the viscosity coefficients of the "extracted" subelement ν. At low frequencies ($\nu = 2$) when

$$\frac{1}{\tau_2^{\varepsilon}(k,l)},\ \frac{1}{\tau_2^{\sigma}(i,j)} \sim f \ll \frac{1}{\tau_3^{\varepsilon}(k,l)},\ \frac{1}{\tau_3^{\sigma}(i,j)};\ \frac{1}{\tau_4^{\varepsilon}(k,l)},\ \frac{1}{\tau_4^{\sigma}(i,j)};\ \ldots$$

(1.17) becomes

$$L_2^{abkl}A_{ijkl}^{\circ 2}\dot{\sigma}_{1\Sigma}^{ij} + \sigma_{1\Sigma}^{ab} = L_2^{abkl}\dot{\varepsilon}_{kl} + N_{1p}^{ab} + L_2^{abkl}A_{ijkl}^{\circ 2}(\dot{N}_{1p}^{ij} + p_{\Sigma}\dot{g}^{ij}) + p_{\Sigma}g^{ab}$$

In principle, equations of the type (1.17) can be used approximately, instead of (1.2), at any frequencies in a fairly narrow range. However, the values of the coefficients therein will depend on the mean frequency f_0 of the range selected. Here L_{ν}^{ijkl} and $A_{mnkl}^{\circ\nu}$ decrease as f_0 increases while $A_{\Sigma 2}^{ijkl}$ increase.

Taking account of the first relationship in (1.2), after eliminating Δ_{kl}^{1} from N_1^{ij} Eqs. (1.1), (1.17), (1.3) yield a connection between the stress and strain tensor components σ^{ij} and ε_{ij} that characterise the muscle tissue as a whole.

Let us present the integral form of the connection between the stress and strain tensor components σ^{ij} and ε_{ij}. We will first write the connection between the components $\sigma_{1\alpha}^{ij}$ and ε_{ij} for a separate subelement α by eliminating Δ_{ij}^{α}. By virtue of (1.5), (1.9), (1.10), the expression for N_1^{ij} can be represented in the form

$$N_1^{ij} = N_{01}^{ij} + N_{\varepsilon 1}^{ijkl}\varepsilon_{kl} + N_{\Delta 1}^{ijkl}\Delta_{kl}^{1}$$

For a subelement with number $\alpha = 1$ we then have from (1.2), taking (1.12) into account

$$\begin{aligned} &C_{\alpha kl}^{ab}\dot{\sigma}_{1\alpha}^{kl} + a_{\alpha mn}^{ab}\dot{\sigma}_{1\alpha}^{mn} = r_{\alpha}^{ab} \\ &C_{\alpha kl}^{ab} = \delta_k^{\ a}\delta_l^{\ b} + N_{\Delta\alpha}^{abij}A_{ijkl}^{\circ\alpha}, \quad a_{\alpha mn}^{ab} = L_{\alpha}^{abkl}A_{klmn}^{\circ\alpha} \\ &r_{\alpha}^{ab} = L_{\alpha}^{abkl}\dot{\varepsilon}_{kl} + (N_{\varepsilon\alpha}^{abkl} + N_{\Delta\alpha}^{abkl})\varepsilon_{kl} + N_{0\alpha}^{ab} + p_{\alpha}'g^{ab} \end{aligned} \tag{1.18}$$

Applying the Laplace transform to (1.18)

$$F_{1\alpha}^{ab}(s) = \int_0^{\infty}\sigma_{1\alpha}^{ab}(t)\,e^{-st}\,dt,\ F_{r\alpha}^{ab}(s) = \int_0^{\infty}r_{\alpha}^{ab}(t)\,e^{-st}dt$$

we obtain

$$\begin{aligned} &F_{1\alpha}^{kl} = b_{\alpha ab}^{*kl}F_{r\alpha}^{ab} + b_{\alpha ab}^{*kl}a_{\alpha mn}^{ab}\sigma_{1\alpha}^{mn}(t=0) \\ &b_{\alpha ab}^{*mn}(C_{\alpha kl}^{ab} + sa_{\alpha kl}^{b}) = \delta_k^{\ m}\delta_l^{\ n} \end{aligned} \tag{1.19}$$

Setting $\sigma_{1\alpha}^{n}(t=0) = 0$ for simplicity, and applying the inverse Laplace transform to (1.19) we obtain

$$\sigma_{1\alpha}^{kl}(t) = \int_0^t f_{\alpha ab}^{kl}(t-\tau)\,r_{\alpha}^{ab}(\tau)\,d\tau \tag{1.20}$$

$$f_{\alpha ab}^{kl}(t) = \frac{1}{2\pi i}\int_{q-i\infty}^{q+i\infty}e^{st}b_{\alpha ab}^{*kl}(s)\,ds \tag{1.21}$$

Summing (1.20) over all α, we find $\sigma_{1\Sigma}^{kl}$ and by using (1.1)-(1.3) we write the desired connection between σ^{ij} and ε_{ij}

$$\sigma^{ij} = pg^{ij} + \varphi_2 B^{ijkl}\varepsilon_{kl} + \varphi_1\int_0^t\varphi^{ijmn}(t-\tau)\,\dot{\varepsilon}_{mn}(\tau)\,d\tau + \tag{1.22}$$

$$\varphi_1\int_0^t f_{1ab}^{ij}(t-\tau)[(N_{\varepsilon 1}^{abmn} + N_{\Delta 1}^{abmn})\varepsilon_{mn}(\tau) + N_{01}^{ab}]\,d\tau$$

$$\varphi^{klmn}(t) = \sum_{\alpha=2}^{\infty}f_{\alpha ab}^{kl}(t)\,L_{\alpha}^{abmn} \tag{1.23}$$

In the case of a continuous distribution of the relaxation time spectrum, the tensor components of the relaxation function $\varphi^{ijmn}(t)$ are not expressed in terms of the sum of the functions (1.21) in the form (1.23). In this sense, (1.22) can be considered to be more general than (1.1)-(1.3).

Within the framework of a model of the type (1.1)-(1.3), (1.6) taking the compressibility into account, expressions are obtained in /1, 2, 7/ for the velocity and damping of longitudinal waves propagating along the across the muscle fibres; here the contribution of the activity N_{α}^{ij} is determined only by the term of the type (1.10). For transverse waves /1/ propagating along the fibres, a term of the type (1.9) in addition to (1.10) affects the velocity and

damping while only (1.10) is effective for perpendicular propagation. Expressions are presented in /6/ for the longitudinal wave velocity and damping in thin muscle specimens obtained within the framework of the model (1.1)-(1.4), (1.16) for $N_\alpha^{ij} \neq 0$, $L_\alpha^{ijkl} \neq 0$ ($\alpha = 1, 2, \ldots$).

2. We now consider a cylindrical specimen of longitudinal section of muscle tissue, described by the equations considered above. The miscle fibres are parallel to the specimen axis. For simplicity we assume the specimen lateral surface to be stress-free. Let the axis $x^3 = z$ of the rectangular Cartesian coordinate system x^i be directed along the specimen axis. We assume the specimen to be fairly thin. Only one stress tensor component σ^{33} is not zero in such a specimen.

Let the specimen be subjected to longitudinal displacements (along the specimen axis). The transverse displacements are also non-zero. It can be shown that $\varepsilon_{11} = \varepsilon_{22}$, $\Delta_{11}{}^\alpha = \Delta_{22}{}^\alpha$, $\eta_{11}{}^\alpha = \eta_{22}{}^\alpha$, $\sigma_1{}^{11} = \sigma_1{}^{22}$, $\sigma_2{}^{11} = \sigma_2{}^{22}$. The incompressibility conditions have the form $2\varepsilon_{11} + \varepsilon_{33} = 0$, $2\Delta_{11}{}^\alpha + \Delta_{33}{}^\alpha = 0$, $2\eta_{11}{}^\alpha + \eta_{33}{}^\alpha = 0$. Then we write the governing Eqs.(1.2), (1.17), (1.3) as follows

$$\sigma_1{}^{11} = p_1 + A_{\Sigma 1}\varepsilon_{33} + \sigma_{1\nu}^{11} + N_{1p}^{11} \tag{2.1}$$
$$\sigma_{1\nu}^{11} = p_\nu' + L_{\nu 1}\Delta_{33}{}^{\cdot\nu} = A_{\nu 1}\eta_{33}{}^\nu, \quad N_{1p}^{11} = A_1\eta_{33}{}^1$$
$$\sigma_1{}^{33} = p_1 + A_{\Sigma 3}\varepsilon_{33} + \sigma_{1\nu}^{33} + N_{1p}^{33} \tag{2.2}$$
$$\sigma_{1\nu}^{33} = p_\nu' + L_{\nu 3}\Delta_{33}{}^{\cdot\nu} = A_{\nu 3}\eta_{33}{}^\nu, \quad N_{1p}^{33} = A_3\eta_{33}{}^1$$
$$\sigma_2{}^{11} = p_2 + B_1\varepsilon_{33}, \quad \sigma_2{}^{33} = p_2 + B_3\varepsilon_{33} \tag{2.3}$$
$$A_{\Sigma 1} = \sum_{\alpha=2}^{\nu-1}(-\mu_A{}^\alpha + \nu_{A2}^\alpha), \quad A_{\nu 1} = -\mu_A{}^\nu + \nu_{A2}^\nu, \quad A_1 = -\mu_A{}^1 + \nu_{A2}^1$$
$$L_{\nu 1} = -\mu_L{}^\nu + \nu_{L2}^\nu, \quad B_1 = -\mu + \nu_2, \quad A_{\Sigma 3} = \sum_{\alpha=2}^{\nu-1}(2\mu_A{}^\alpha + 4\nu_{A1}^\alpha + \nu_{A2}^\alpha + \nu_{A3}^\alpha)$$
$$A_{\nu 3} = 2\mu_A{}^\nu + 4\nu_{A1}^\nu + \nu_{A2}^\nu + \nu_{A3}^\nu, \quad A_3 = 2\mu_A{}^1 + 4\nu_{A1}^1 + \nu_{A2}^1 + \nu_{A3}^1$$
$$L_{\nu 3} = 2\mu_L{}^\nu + 4\nu_{L1}^\nu + \nu_{L2}^\nu + \nu_{L3}^\nu, \quad B_3 = 2\mu + 4\nu_1 + \nu_2 + \nu_3$$

Taking account of (2.1) and (2.3) the boundary conditions on the specimen lateral surface results in the relationship

$$\varphi_1(p_1 + A_{\Sigma 1}\varepsilon_{33} + A_{\nu 1}\eta_{33}{}^\nu + A_1\eta_{33}{}^1) + \varphi_2(p_2 + B_1\varepsilon_{33}) = 0 \tag{2.4}$$

We have the following expression for σ^{33}

$$\sigma^{33} = \varphi_1(p_1 + A_{\Sigma 3}\varepsilon_{33} + A_{\nu 3}\eta_{33}{}^\nu + A_3\eta^1{}_{33}') + \varphi_2(p_2 + B_3\varepsilon_{33}) \tag{2.5}$$

In the case of small deformations the relationships

$$N_1{}^{33} = n_1{}^{33} + m_1{}^{33} = \gamma + \gamma'\Delta_{33}{}^1 + \gamma\varepsilon_{33} + \Gamma_3\Delta_{33}{}^1 \tag{2.6}$$
$$N_1{}^{11} = m_1{}^{11} = \Gamma_1\Delta_{33}{}^1 \tag{2.7}$$
$$(\gamma \equiv \gamma_1, \ \Gamma_3 = 2\Gamma_\mu + 4\Gamma_{\nu 1} + \Gamma_{\nu 2} + \Gamma_{\nu 3}, \ \Gamma_1 = -\Gamma_\mu + \Gamma_{\nu 2})$$

follow from (1.5), (1.9), (1.10) for the active stresses.

Eliminating $\Delta_{33}{}^\nu$, $\Delta_{33}{}^1$, $\varphi_1 p_1 + \varphi_2 p_2$ from (2.4)-(2.7) and substituting into $\sigma^{33} = \varphi_1\sigma_1{}^{33} + \varphi_2\sigma_2{}^{33}$ using (2.2) and (2.3), we obtain an equation connecting σ^{33} and ε_{33}

$$L_{\nu p}\sigma^{\cdot 33} + \sigma^{33} = E\varepsilon_{33} + L_{\nu p}(E + \varphi_1 E_\nu)\varepsilon_{33}{}^\cdot + \gamma_0 \tag{2.8}$$
$$L_{\nu p} = \frac{L_\nu}{E_\nu}, \quad E = \varphi_2 E_2 + \varphi_1 E_\Sigma + \varphi_1 E_1 \frac{\gamma + \gamma' + E_\Gamma}{E_1 + \gamma' + E_\Gamma}, \quad \gamma_0 = \frac{\gamma\varphi_1 E_1}{E_1 + \gamma' + E_\Gamma}$$
$$E_\nu = A_{\nu 3} - A_{\nu 1}, \quad E_1 = A_3 - A_1, \quad E_2 = B_3 - B_1$$
$$E_\Sigma = A_{\Sigma 3} - A_{\Sigma 1}, \quad E_\Gamma = \Gamma_3 - \Gamma_1, \quad L_\nu = L_{\nu 3} - L_{\nu 1}$$

The coefficients in (2.8) are constant in time. The quantity E_2 is Young's modulus of the elastic passive phase 2, E_1, E_ν, E_Σ are Young's moduli of the elastic elements in the active phase 1, E_Γ is Young's modulus of the microrelations, L_ν is the coefficient of viscosity of phase 1, and γ_0 has the meaning of the active muscle stress.

3. As an illustration of the wave process we investigate longitudinal waves in a specimen of muscle tissue whose transverse linear dimension is small compared with the wavelength. The wave processes occur in a certain state of stress and strain (background) that occurs after activation of the muscle. We present the relationships just for the waves.

Using the equations of motion and the relationship between the strain ε_{33} and the displacement $w_3 = w^3$

$$\rho\frac{\partial^2 w_3}{\partial t^2}=\frac{\partial\sigma^{33}}{\partial z},\quad \varepsilon_{33}=\frac{\partial w_3}{\partial z},\ \rho v_0^2=E$$

we obtain an equation to investigate the waves in the form

$$\rho L_{\nu p}\frac{\partial^3 w_3}{\partial t^3}+\rho\frac{\partial^2 w_3}{\partial t^2}=\rho v_0^2\frac{\partial^2 w_3}{\partial z^2}+L_{\nu p}(\rho v_0^2+\varphi_1 E_\nu)\frac{\partial^3 w_3}{\partial t\,\partial z^2} \tag{3.1}$$

using (2.8).

Formally it is no different from the well-known equation in the theory of linear viscoelasticity /12/. When investigating the solution, of fundamental interest here is its dependence on the active parameters γ, γ', E_Γ, which can change depending on the state of the muscle tissue since it is a medium with variable structure.

If the wave frequency f is large compared with the characteristic frequencies τ_c^{-1} of muscle contraction, then the coefficents in (3.1) can be slow functions of time (for instance, for a single background contraction), where $\gamma'=0$. Conversely, if $f\ll\tau_c^{-1}$ it is necessary to set $E_\Sigma=0$, $\nu=2$ and $\gamma'\neq 0$ in (3.1) for a certain background activated state that is constant in time. For $f\sim\tau_c^{-1}$ the model requires refinement.

Substituting $w_3=w_0\exp[i(\omega t-kz)]$ into (3.1), where w_0 is the constant wave amplitude, we obtain the dispersion equation

$$k^2=\frac{\omega^2(1+iL_{\nu_p}\omega)}{v_0^2(1+iaL_{\nu_p}\omega)},\quad a=\frac{\rho v_0^2+\varphi_1 E_\nu}{\rho v_0^2} \tag{3.2}$$

Considering the frequency ω as real (given), we use (3.2) to find the velocities $v_{1,2}$ and damping factors $\varkappa_{1,2}$ of the longitudinal waves propagating to opposite sides (the muscle specimen is considered to be infinite in the longitudinal direction)

$$\begin{gathered}w_3=w_0\exp[i(\omega t\mp k_d z)]\exp(\pm k_m z),\ k=k_d+ik_m\\ k_d=[(n^2+m^2)^{1/2}+n]^{1/2},\quad v_{1,2}=\pm\frac{\omega}{k_d},\quad \varkappa_{1,2}=\mp k_m=\mp\frac{m}{k_d}\\ m=\frac{L_{\nu p}\omega k_0^2(1-a)}{2[1+(L_{\nu p}\omega a)^2]},\quad n=mL_{\nu p}\omega a+\frac{k_0^2}{2},\quad k_0^2=\frac{\omega^2}{v_0^2}\end{gathered} \tag{3.3}$$

If the damping is neglected ($L_{\nu_p}\omega\ll 1$) then for $v_0^2>0$ we obtain from (3.3) an expression for the longitudinal wave propagation velocities $v_{1,2}$ in the form

$$v_{1,2}=\pm(v_0^2)^{1/2}=\pm\frac{\Lambda^{1/2}}{\rho^{1/2}},\quad \Lambda=\varphi_2 E_2+\varphi_1 E_\Sigma+\varphi_1 E_1\frac{\gamma+\gamma'+E_\Gamma}{E_1+\gamma'+E_\Gamma} \tag{3.4}$$

It follows from (3.4) that the propagation velocity depends not only on Young's moduli E_2, E_1, E_Σ of the phase passive elements (which can, however, depend on the muscle state), but also on Young's modulus E_Γ of the microrelations occurring because of the biochemical reactions after muscle excitation, and also on the activity parameters: γ, characterizing the active stress, and γ', which depends on the strain in the background state. If it is assumed, to be specific, that the additional elastic coefficients originating because of the anisotropy $\nu_{1,2,3}$, $\nu^\alpha_{A1,2,3}$, $\nu^\alpha_{L1,2,3}$, $\Gamma_{\nu_{1,2,3}}$ are small compared with the ordinary isotropic elastic Lamé coefficients μ, $\mu_A{}^\alpha$, $\mu_L{}^\alpha$, Γ_μ, respectively, then E_2, E_1, E_ν, E_Σ, E_Γ, L_ν are positive. The quantity $\gamma\geqslant 0$, while γ' can be positive, zero, or negative.

Knowing $\varphi_{1,2}$ and also the viscoelastic and active parameters that can be determined in static and dynamic experiments on a muscle tissue specimen the velocity and damping of this wave can be calculated. Conversely, knowing the wave velocity and damping in different muscle states, certain viscoelastic and active tissue parameters, or at least the connection between them, can be determined.

It follows from (3.3) that $m<0$ for all reasonable values of the parameters, i.e., the longitudinal waves will be attenuated as they propagate.

4. We will consider the question of the stability in this system. Using the notation $\omega=-is_1$ we obtain from (3.2) that

$$L_{\nu_1}s_1^3+s_1^2+L_{\nu p}av_0^2k^2s_1+v_0^2k^2=0 \tag{4.1}$$

For $\omega_m>0$ the quantity $\omega=\omega_d+i\omega_m$ yields stability in time; the real part of s_1 is negative here. For real k the negativity condition of the real parts of the roots of (4.1) reduce to the following, according to the Routh-Hurwitz criterion:

$$v_0^2\geqslant 0 \tag{4.2}$$

It follows from (4.2) that the system is stable if

$$\Lambda \geqslant 0 \tag{4.3}$$

Since there are no branch points of the function $k_{1,2}(\omega)$ from (3.2) in the lower ω half-plane $(\omega_m < 0)$, then if (4.3) is not satisfied the instability will be convective /13/.

Let the specimen of muscle tissue have the finite length L. Let us examine the special case of rigid clamping of the specimen when the boundary conditions reduce to the following:

$$w_3(z=0) = w_3(z=L) = 0 \tag{4.4}$$

Using the particular solution of the wave equation in the form

$$w_3(z,t) = e^{i\omega t}(C_1 e^{-ikz} + C_2 e^{ikz})$$

where $k = k(\omega)$ is found from the dispersion Eq.(3.2), and the boundary conditions (4.4), we obtain an equation for the complex frequency

$$iL_{\nu_1}\omega^3 - \omega^2 - iL_{\nu_1} a v_0^2 \frac{\pi^2 n^2}{L^2}\omega - \frac{\pi^2 n^2}{L^2} v_0^2 = 0 \tag{4.5}$$

For $\omega = -is_1$ Eq.(4.5) is analogous to (4.1) and its analysis yields a stability condition that agrees with (4.2) for $n^2 > 0$. For $n = 0$ the stability condition is always satisfied, as follows from (4.5).

Therefore, for a muscle tissue specimen of finite length L violation of condition (4.3) leads to absolute instability, but only to convective instability for a specimen of infinite length.

Instability, that is possible, in principle, at low frequencies $f \ll \tau_c^{-1}$ for $\gamma' < 0$, is determined from one of the following conditions as follows from (4.3):

$$-E_1 - E_\Gamma < \gamma' < -E_1 - E_\Gamma + \frac{E_1 - \gamma}{1 + \varphi_2 E_2/\varphi_1 E_1} \quad (\gamma < E_1)$$
$$-E_1 - E_\Gamma + \frac{E_1 - \gamma}{1 + \varphi_2 E_2/\varphi_1 E_1} < \gamma' < -E_1 - E_\Gamma \quad (\gamma > E_1)$$

We note that if $N_\alpha^{ij} \neq 0$, $L_\alpha^{ijkl} \neq 0$ $(\alpha = 1, 2, \ldots)$, where the expressions for N_α^{ij} have a structure of the type (1.9), (1.10), then satisfaction of the system instability conditions is facilitated (in principle, there may be instability even for $\gamma_\alpha' > 0$ because of the fairly large quantity γ_α). Wave amplification is possible.

When there is a loss of stability (conditions (4.3) are not satisfied), the muscle tissue specimen possibly becomes a generator of selfexcited oscillations whose parameters can be found when taking account of the non-linear properties of the muscle. The selfexcited oscillations observed experimentally in muscle specimens are described in /14/, for example.

5. We will compare expressions obtained for the velocities and damping (3.3) and (3.4) with the results of experiments /10, 15-17/.

Initially, we will recall briefly the characteristic features of the muscle tissue microstructure /18/. The muscle tissue fibres at the microlevel are formed parallel to the arranged myofibrillars, which are, in turn, filled with protein microfilaments of two kinds, active (*a*) (thin) and myosin (*m*) (thick, Fig.2). Muscle contraction is accompanied by insertion of some filaments between others with closing and opening of the microconnections (*c*) (bridges) between the filaments taking part. The muscle can be in different states: passive (bridges open), a single contraction (in response to a single stimulating signal), tetanic contraction (periodic excitation at high frequency), and rigidity or contracture (bridges closed and fixed). The mechanical characteristics (elasticity and viscosity, say) of the filaments and bridges here generally depend on the state of the muscle tissue.

Let the stability conditions (4.3) be satisfied. In this case the expression for the longitudinal wave propagation velocity in the muscle tissue specimen (3.4) will yield a monotonic increase in the velocity as the active parameter γ or γ', or E_Γ increases.

We examine the simplest special case when either the wave frequency is $f \gg \tau_c^{-1}$ (for skeletal muscles $\tau_c \sim 10^{-2}$ sec), or muscle contraction occurs in the background state within the limits of the "plateau" of the bell-shaped dependence $\gamma = \gamma(l_\Delta)$. In both these cases $\gamma' = 0$. We use for the estimates /5, 10, 14/

$$E_\nu = (1+\beta)E_{\nu 0}, \quad E_\Sigma = (1+\beta)E_{\Sigma 0}, \quad L_{\nu_1} = L_{\nu_1 0}/(1+\beta_1)$$
$$E_{\nu 0} = E_1/0{,}6, \quad \varphi_2 E_2 = 10^5 \text{ N/m}2, \quad \varphi_1 E_{\Sigma 0} = 10^6 \text{ N/m}^2$$
$$\varphi_1 E_1 = 1.2\cdot 10^7 \text{ N/m}^2, \quad L_{\nu_1 0} = 5\cdot 10^{-6}\text{sec}, f = 3\cdot 10^3 \text{ Hz}, \quad \rho = 1035 \text{ kg/m}^3$$

For a passive muscle $(\gamma = \beta = \beta_1 = E_\Gamma = 0)$ we have $v_1 = 52{,}9$ m/sec and $\varkappa_1 = 188$ m^{-1} from (3.3). For a muscle in the rigid state (the muscle is passive but all the bridges are closed), we find $v_1 = 144$ m/sec, $\varkappa_1 = 42$ m^{-1} by setting $\gamma = 0$, $\beta = 8$, $\beta_1 = 0.5$, $E_\Gamma = E_1$. If the muscle is in the activated state (tetanus), then we take $\varphi_1\gamma = 7.5\cdot 10^5$ N/m^2 $\beta = 14$, $\beta_1 = 0.7$, $E_\Gamma = 1{,}5\,E_1$; and we have $v_1 = 175$ m/sec and $\varkappa_1 = 34$ m^{-1}.

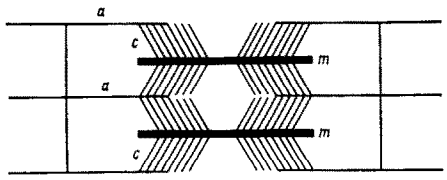

Fig.2

It can be shown from a comparison with test data /10, 15-17/ that these estimates of the values of the longitudinal wave velocities and damping in a thin muscle tissue specimen in the passive, rigid, and tetanic states are close to those observed.

If the wave frequency f is comparable with the natural characteristic frequencies of muscle contraction τ_c^{-1}, then resonance effects can exist, and parametric phenomena can apparently occur. Internal parameters describing the cyclical structural rebuilding of proteins and their equations must be introduced to describe oscillatory processes at the muscle tissue microlevel.

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